Angular Momentum Linear Inverted Pendulum

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The Angular momentum Linear Inverted Pendulum (ALIP) model can be used to describe a point mass moving along a horizontal line at a fixed height. It differs from convention Linear Inverted Pendulum (LIP) model by using angular momentum about the ground contact point L instead of Center of Mass (CoM) horizontal velocity v as the second system state variable in addition to CoM horizontal position x.

The Equations of Motion (EoM) of ALIP for the single support continuous phase can be derived as

$$\begin{cases} \dot{x} = \dfrac{L}{mH}, \\ \dot{L} = mgx, \end{cases} \tag{1}$$

by assuming constant CoM height H and point body mass m, where x and $\dot{x}$ represent the position and velocity of CoM along the x axis, respectively.

The double support impact phase of ALIP can be described as

$$L_2 = L_1 + \boldsymbol{p}_{2\to 1} \times m\boldsymbol{v} \tag{2}$$

where the old support leg is labeled 1, the new support leg is labeled 2, $\boldsymbol{p}_{2\to 1}$ is the vector from the new support point to the old support point, and $\boldsymbol{v} = [\dot{x}, \dot{z}]^T$. When assuming flat ground and zero vertical CoM velocity, the reset map can be defined as

$$\begin{cases} x(t_{k+1}^+) = p_x^{\text{sw}\to\text{CoM}}(t_k^-), \\ L(t_{k+1}^+) = L(t_k^-) \end{cases} \tag{3}$$

where t_{k+1}^+ indicates the time instant just after the k-th impact, t_k^- indicates the time instant just before the k-th impact, $p_x^{\text{sw}\to\text{CoM}}$ represents the projection of the vector from swing foot (new ground contact point) to CoM along the x axis.

The ALIP model in the single support phase can be analytically solved for a closed-form solution, which is similar to the process of solving the LIP model. The solution is

$$\begin{cases} x(t) = x(t_o)\cosh[\lambda(t-t_o)] + \dfrac{L(t_o)}{mH\lambda}\sinh[\lambda(t-t_o)], \\ L(t) = x(t_o)mH\lambda\sinh[\lambda(t-t_o)] + L(t_o)\cosh[\lambda(t-t_o)]. \end{cases} \tag{4}$$

Now let us develop the control law to look one step ahead. At any time instant t within the k-th step, the angular momentum at the end of this step, t_k^-, can be estimated as

$$\hat{L}(t_k^-) = x(t)mH\lambda\sinh[\lambda(t_k^- - t)] + L(t)\cosh[\lambda(t_k^- - t)]. \tag{5}$$

Then the system goes through the reset map for impact, and the states become

$$\begin{cases} x(t_{k+1}^+) = p_x^{\text{sw}\to\text{CoM}}(t_k^-), \\ \hat{L}(t_{k+1}^+) = \hat{L}(t_k^-). \end{cases} \tag{6}$$

Therefore, the angular momentum at the end of the next step, t_{k+1}^-, can be estimated as

$$\begin{aligned}\hat{L}(t_{k+1}^-) &= x(t_{k+1}^+)mH\lambda\sinh[\lambda(t_{k+1}^- - t_{k+1}^+)] + \hat{L}(t_{k+1}^+)\cosh[\lambda(t_{k+1}^- - t_{k+1}^+)]\\ &= p_x^{\text{sw}\to\text{CoM}}(t_k^-)mH\lambda\sinh(\lambda T_{k+1}) + \hat{L}(t_k^-)\cosh(\lambda T_{k+1})\\ &= p_x^{\text{sw}\to\text{CoM}}(t_k^-)mH\lambda\sinh(\lambda T_{k+1}) + \{x(t)mH\lambda\sinh[\lambda(t_k^- - t)] + L(t)\cosh[\lambda(t_k^- - t)]\}\cosh(\lambda T_{k+1}),\end{aligned} \tag{7}$$

where T_{k+1} is the step period of the $(k+1)$-th step. If set $\hat{L}(t_{k+1}^-)$ as the control target, *i.e.* $\hat{L}(t_{k+1}^-) = L_{y\text{des}}$, then $p_x^{\text{sw}\to\text{CoM}}(t_k^-)$ becomes the control means that can be adjusted based on

$$\begin{aligned}p_{x\text{des}}^{\text{sw}\to\text{CoM}}(t_k^-) &= \frac{L_{y\text{des}} - \hat{L}(t_k^-)\cosh(\lambda T_{k+1})}{mH\lambda\sinh(\lambda T_{k+1})}\\ &= \frac{L_{y\text{des}} - \{x(t)mH\lambda\sinh[\lambda(t_k^- - t)] + L(t)\cosh[\lambda(t_k^- - t)]\}\cosh(\lambda T_{k+1})}{mH\lambda\sinh(\lambda T_{k+1})}\end{aligned} \tag{8}$$

Note that the above derivation focuses on the movement along x-axis, but can also be applied to y-axis for lateral movement. Consider gaits of stepping in place with step width W and no flight phase, due to symmetry, $p_{y\text{des}}^{\text{sw}\to\text{CoM}} = \pm W/2$ and $\hat{L}(t_{k+1}^-) = -\hat{L}(t_k^-) = L_{x\text{des}}$, thus

$$\begin{aligned}\pm\frac{W}{2} = p_{y\text{des}}^{\text{sw}\to\text{CoM}}(t_k^-) &= \frac{L_{x\text{des}} - \hat{L}(t_k^-)\cosh(\lambda T_{k+1})}{mH\lambda\sinh(\lambda T_{k+1})} = L_{x\text{des}}\frac{1+\cosh(\lambda T_{k+1})}{mH\lambda\sinh(\lambda T_{k+1})}\\ \Rightarrow L_{x\text{des}} &= \pm\frac{1}{2}mHW\frac{\lambda\sinh(\lambda T_{k+1})}{1+\cosh(\lambda T_{k+1})}\end{aligned} \tag{9}$$

where the plus or minus sign is determined by the coming support leg, either left or right.

The stability of ALIP can be analyzed using the Poincaré map. Take the moment just after impact as the Poincaré section, we can write the Poincaré map along the x axis as

$$\begin{aligned}\begin{bmatrix}x(t_k^+)\\ L(t_k^+)\end{bmatrix} &= \begin{bmatrix}p_{x\text{des}}^{\text{sw}\to\text{CoM}}(t_{k-1}^-)\\ L(t_{k-1}^-)\end{bmatrix}\\ &= \begin{bmatrix}\dfrac{L(t_k^-) - [x(t_{k-1}^+)mH\lambda\sinh(\lambda T_{k-1}) + L(t_{k-1}^+)\cosh(\lambda T_{k-1})]\cosh(\lambda T_k)}{mH\lambda\sinh(\lambda T_k)}\\ x(t_{k-1}^+)mH\lambda\sinh(\lambda T_{k-1}) + L(t_{k-1}^+)\cosh(\lambda T_{k-1})\end{bmatrix}\\ &= \begin{bmatrix}-\dfrac{x(t_{k-1}^+)\sinh(\lambda T_{k-1})\cosh(\lambda T_k)}{\sinh(\lambda T_k)} - \dfrac{L(t_{k-1}^+)\cosh(\lambda T_{k-1})\cosh(\lambda T_k)}{mH\lambda\sinh(\lambda T_k)} + \dfrac{L(t_k^-)}{mH\lambda\sinh(\lambda T_k)}\\ x(t_{k-1}^+)mH\lambda\sinh(\lambda T_{k-1}) + L(t_{k-1}^+)\cosh(\lambda T_{k-1})\end{bmatrix}\\ &= \begin{bmatrix}-\dfrac{\sinh(\lambda T_{k-1})\cosh(\lambda T_k)}{\sinh(\lambda T_k)} & -\dfrac{\cosh(\lambda T_{k-1})\cosh(\lambda T_k)}{mH\lambda\sinh(\lambda T_k)}\\ mH\lambda\sinh(\lambda T_{k-1}) & \cosh(\lambda T_{k-1})\end{bmatrix}\begin{bmatrix}x(t_{k-1}^+)\\ L(t_{k-1}^+)\end{bmatrix} + \begin{bmatrix}\dfrac{L(t_k^-)}{mH\lambda\sinh(\lambda T_k)}\\ 0\end{bmatrix}\end{aligned} \tag{10}$$

Define $T_k = T_{k-1} = T$ for fixed periodic gaits, the Poincaré map becomes

$$\begin{aligned}\begin{bmatrix}x(t_k^+)\\ L(t_k^+)\end{bmatrix} &= \begin{bmatrix}-\dfrac{\sinh(\lambda T_{k-1})\cosh(\lambda T_k)}{\sinh(\lambda T_k)} & -\dfrac{\cosh(\lambda T_{k-1})\cosh(\lambda T_k)}{mH\lambda\sinh(\lambda T_k)}\\ mH\lambda\sinh(\lambda T_{k-1}) & \cosh(\lambda T_{k-1})\end{bmatrix}\begin{bmatrix}x(t_{k-1}^+)\\ L(t_{k-1}^+)\end{bmatrix} + \begin{bmatrix}\dfrac{L(t_k^-)}{mH\lambda\sinh(\lambda T_k)}\\ 0\end{bmatrix}\\ &= \begin{bmatrix}-\cosh(\lambda T) & -\dfrac{\cosh^2(\lambda T)}{mH\lambda\sinh(\lambda T)}\\ mH\lambda\sinh(\lambda T) & \cosh(\lambda T)\end{bmatrix}\begin{bmatrix}x(t_{k-1}^+)\\ L(t_{k-1}^+)\end{bmatrix} + \begin{bmatrix}\dfrac{L(t_k^-)}{mH\lambda\sinh(\lambda T)}\\ 0\end{bmatrix}\end{aligned} \tag{11}$$

The corresponding fixed point can be obtained by setting $x(t_k^+) = x(t_{k-1}^+) = x^*$ and $L(t_k^+) = L(t_{k-1}^+) = L^*$,

$$\begin{aligned}
&\begin{bmatrix} 1+\cosh(\lambda T) & \dfrac{\cosh^2(\lambda T)}{mH\lambda\sinh(\lambda T)} \\ -mH\lambda\sinh(\lambda T) & 1-\cosh(\lambda T) \end{bmatrix} \begin{bmatrix} x^* \\ L^* \end{bmatrix} = \begin{bmatrix} \dfrac{L(t_k^-)}{mH\lambda\sinh(\lambda T)} \\ 0 \end{bmatrix} \\
\Rightarrow &\begin{bmatrix} x^* \\ L^* \end{bmatrix} = \begin{bmatrix} 1+\cosh(\lambda T) & \dfrac{\cosh^2(\lambda T)}{mH\lambda\sinh(\lambda T)} \\ -mH\lambda\sinh(\lambda T) & 1-\cosh(\lambda T) \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{L(t_k^-)}{mH\lambda\sinh(\lambda T)} \\ 0 \end{bmatrix} \\
\Rightarrow &\begin{bmatrix} x^* \\ L^* \end{bmatrix} = \begin{bmatrix} 1-\cosh(\lambda T) & -\dfrac{\cosh^2(\lambda T)}{mH\lambda\sinh(\lambda T)} \\ mH\lambda\sinh(\lambda T) & 1+\cosh(\lambda T) \end{bmatrix} \begin{bmatrix} \dfrac{L(t_k^-)}{mH\lambda\sinh(\lambda T)} \\ 0 \end{bmatrix} \\
\Rightarrow &\begin{bmatrix} x^* \\ L^* \end{bmatrix} = \begin{bmatrix} \dfrac{1-\cosh(\lambda T)}{mH\lambda\sinh(\lambda T)} L(t_k^-) \\ L(t_k^-) \end{bmatrix}
\end{aligned} \tag{12}$$

The corresponding eigenvalues are calculated to be two zeros,

$$\begin{vmatrix} -\cosh(\lambda T) - \Lambda & -\dfrac{\cosh^2(\lambda T)}{mH\lambda\sinh(\lambda T)} \\ mH\lambda\sinh(\lambda T) & \cosh(\lambda T) - \Lambda \end{vmatrix} = \Lambda^2 = 0 \Rightarrow \Lambda_1 = \Lambda_2 = 0 \tag{13}$$

Instead of the deadbeat control in (8), an adaptive control law can be designed using a coefficient $\alpha \in (-1, 1)$ as

$$L_{y\text{des}} - \hat{L}(t_{k+1}^-) = \alpha[L_{y\text{des}} - \hat{L}(t_k^-)] \to 0, \tag{14}$$

which results in

$$\begin{aligned}
p_{x\text{des}}^{\text{sw}\to\text{CoM}}(t_k^-) &= \frac{L_{y\text{des}} - \alpha[L_{y\text{des}} - \hat{L}(t_k^-)] - \hat{L}(t_k^-)\cosh(\lambda T_{k+1})}{mH\lambda\sinh(\lambda T_{k+1})} \\
&= \frac{L_{y\text{des}} - \alpha L_{y\text{des}}}{mH\lambda\sinh(\lambda T_{k+1})} + \frac{\alpha\hat{L}(t_k^-) - \hat{L}(t_k^-)\cosh(\lambda T_{k+1})}{mH\lambda\sinh(\lambda T_{k+1})} \\
&= \frac{1-\alpha}{mH\lambda\sinh(\lambda T_{k+1})} L_{y\text{des}} + \frac{\alpha - \cosh(\lambda T_{k+1})}{mH\lambda\sinh(\lambda T_{k+1})} \hat{L}(t_k^-)
\end{aligned} \tag{15}$$

Note that this controlled system has the same fixed point as the uncontrolled one (the proof is omitted here), but one of the zero eigenvalues becomes α. Since $\alpha \in (-1, 1)$, the system is still asymptotically stable, while the deadbeat control with $\alpha = 0$ is a special case.