Hybrid Dynamics

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Hybrid dynamics is used to describe the evolution of system states in legged locomotion to account for the intermittent ground contacts. This kind of dynamics consists of several phases. For example, in a walking gait, there would be stance phase and impact phase, whereas in a running gait, there would be an additional flight phase. The impact phase happens when a new leg touches ground and it is often treated as an instantaneous event.

1 STANCE PHASE

1.1 Fixed base method

During the stance phase, if using the fixed base methodology, the foot is usually assumed to be pinned down to the ground and not slipping. As a result, the system can be treated as an open kinematic chain. The system dynamics can then be written as

$$\boldsymbol{D}_s(\boldsymbol{q}_s)\ddot{\boldsymbol{q}}_s + \boldsymbol{C}_s(\boldsymbol{q}_s, \dot{\boldsymbol{q}}_s)\dot{\boldsymbol{q}}_s + \boldsymbol{G}_s(\boldsymbol{q}_s) = \boldsymbol{B}_s(\boldsymbol{q}_s)\boldsymbol{u}_s, \tag{1}$$

where $\boldsymbol{q}_s$ is an $N \times 1$ column vector containing all system generalized coordinates and $\boldsymbol{u}_s$ is an $M \times 1$ column vector representing the control input. Usually, the relationship $M < N$ exists, indicating that the system is underactuated. There are also plenty of fully actuated systems, with $M = N$, but the control of it is a bit boring.

According to the Lagrange's Equation, it can be obtained that

$$\boldsymbol{D}_s(\boldsymbol{q}_s) = \frac{\partial}{\partial \dot{\boldsymbol{q}}_s}[\frac{\partial K_s(\boldsymbol{q}_s, \dot{\boldsymbol{q}}_s)}{\partial \dot{\boldsymbol{q}}_s}]^T,$$

$$\boldsymbol{C}_s(\boldsymbol{q}_s, \dot{\boldsymbol{q}}_s) = \frac{\partial[\boldsymbol{D}_s(\boldsymbol{q}_s)\dot{\boldsymbol{q}}_s]}{\partial \boldsymbol{q}_s} - \frac{1}{2}[\frac{\partial \boldsymbol{D}_s(\boldsymbol{q}_s)}{\partial \boldsymbol{q}_s}\dot{\boldsymbol{q}}_s]^T, \tag{2}$$

$$\boldsymbol{G}_s(\boldsymbol{q}_s) = [\frac{\partial V_s(\boldsymbol{q}_s)}{\partial \boldsymbol{q}_s}]^T, \tag{3}$$

$$\boldsymbol{B}_s(\boldsymbol{q}_s) = \left(\frac{\partial}{\partial \boldsymbol{q}_s} \begin{bmatrix} \gamma_1^{rel}(\boldsymbol{q}_s) \\ \gamma_2^{rel}(\boldsymbol{q}_s) \\ \gamma_3^{rel}(\boldsymbol{q}_s) \\ \vdots \\ \gamma_M^{rel}(\boldsymbol{q}_s) \end{bmatrix} \right)^T . \tag{4}$$

where $K_s(\boldsymbol{q}_s, \dot{\boldsymbol{q}}_s)$ and $V_s(\boldsymbol{q}_s)$ are kinetic and potential energy, respectively, and $\gamma_i^{rel}(\boldsymbol{q}_s)$, $i = 1, 2, 3, ..., M$, is the relative displacement corresponding to each entry inside the actuation vector $\boldsymbol{u}_s$. Consequently, it is usually in the form

$$\boldsymbol{B}_s(\boldsymbol{q}_s) = \begin{bmatrix} \boldsymbol{I}_{M \times M} \\ \boldsymbol{0}_{(N-M) \times M} \end{bmatrix}, \tag{5}$$

where q_{M+1}, q_{M+2}, q_{M+3}, ..., q_N of $\boldsymbol{q}_s$ are assumed to be the unactuated degrees of freedom.

The system dynamics can also be written in state space as

$$\dot{\boldsymbol{x}}_s = \begin{bmatrix} \dot{\boldsymbol{q}}_s \\ \boldsymbol{D}_s^{-1}[-\boldsymbol{C}_s(\boldsymbol{q}_s, \dot{\boldsymbol{q}}_s)\dot{\boldsymbol{q}}_s - \boldsymbol{G}_s(\boldsymbol{q}_s) + \boldsymbol{B}_s(\boldsymbol{q}_s)\boldsymbol{u}_s] \end{bmatrix} = \boldsymbol{f}_s(\boldsymbol{x}_s) + \boldsymbol{g}_s(\boldsymbol{x}_s)\boldsymbol{u}_s, \tag{6}$$

which is control affine. The state $\boldsymbol{x}_s$ is defined as $[\boldsymbol{q}, \dot{\boldsymbol{q}}_s]^T$.

1.2 Floating base method

In the floating base context, the Equations of Motion for stance phase can be written as

$$\boldsymbol{D}_e(\boldsymbol{q}_e)\ddot{\boldsymbol{q}}_e + \boldsymbol{C}_e(\boldsymbol{q}_e, \dot{\boldsymbol{q}}_e)\dot{\boldsymbol{q}}_e + \boldsymbol{G}_e(\boldsymbol{q}_e) = \boldsymbol{B}_e(\boldsymbol{q}_e)\boldsymbol{u}_e + \boldsymbol{F}_e, \tag{7}$$

where $\boldsymbol{q}_e$ is a column vector containing all system states, the dimension of which is $(N+L) \times 1$. $\boldsymbol{q}_e$ is an augmented vector based on the generalized coordinates $\boldsymbol{q}_s$ in the fixed base method, *i.e.*

$$\boldsymbol{q}_e = \begin{bmatrix} \boldsymbol{q}_s \\ \boldsymbol{p}_e \end{bmatrix}, \tag{8}$$

where $\boldsymbol{p}_e$, a vector of dimension $L \times 1$, contains the additional generalized coordinates of a fixed point inside the system. The generalized force $\boldsymbol{F}_e$ in Equation (7) is the effective force due to the ground reaction force at stance foot. Therefore, Equation (7) in total has $N+L$ equations and $2N+2L$ unknowns.

The number of unknowns can be reduced by mapping the generalized force $\boldsymbol{F}_e$ to the ground reaction force and torque at the stance foot, $\boldsymbol{F}_f$, as

$$\boldsymbol{F}_e = \boldsymbol{E}^T \boldsymbol{F}_f, \tag{9}$$

where $\boldsymbol{F}_f$ is an $L \times 1$ column vector, and $\boldsymbol{E}$ is the corresponding Jacobian matrix which can be further derived as follows. The position (and pose) of the stance foot can be written as

$$\boldsymbol{p}_f = \boldsymbol{p}_e + \boldsymbol{\gamma}_f(\boldsymbol{q}_s), \tag{10}$$

which can then yield

$$\dot{\boldsymbol{p}}_f = \dot{\boldsymbol{p}}_e + \frac{\partial \boldsymbol{\gamma}_f(\boldsymbol{q}_s)}{\partial \boldsymbol{q}_s}\dot{\boldsymbol{q}}_s = [\frac{\partial \boldsymbol{\gamma}_f(\boldsymbol{q}_s)}{\partial \boldsymbol{q}_s}, \boldsymbol{I}_{L\times L}] \begin{bmatrix} \dot{\boldsymbol{q}}_s \\ \dot{\boldsymbol{p}}_e \end{bmatrix} = [\frac{\partial \boldsymbol{\gamma}_f(\boldsymbol{q}_s)}{\partial \boldsymbol{q}_s}, \boldsymbol{I}_{L\times L}]\dot{\boldsymbol{q}}_e. \tag{11}$$

We can set

$$\boldsymbol{E} = [\frac{\partial \boldsymbol{\gamma}_f(\boldsymbol{q}_s)}{\partial \boldsymbol{q}_s}, \boldsymbol{I}_{L\times L}], \tag{12}$$

then according to power conservation, $\boldsymbol{F}_e \dot{\boldsymbol{q}}_e = \boldsymbol{F}_f \dot{\boldsymbol{p}}_f$, we can arrive at Equation (9) to reduce the number of unknowns in Equation (7) to $N+2L$. On the other hand, because the stance foot is not assumed to be slipping or debouncing during the stance phase, there can be another L equations to make the number of equations and unknowns even,

$$\ddot{\boldsymbol{p}}_f = \boldsymbol{E}\ddot{\boldsymbol{q}}_e + \dot{\boldsymbol{E}}\dot{\boldsymbol{q}}_e = \boldsymbol{0}_{L\times 1}. \tag{13}$$

Combining Equations (7), (9) and (13), the Equations of Motion for the stance phase can be obtained.

2 IMPACT PHASE

2.1 Floating base method

The impact phase performs an instantaneous switch of the stance leg, *i.e.* the swing leg from the previous stance phase becomes the new stance leg during the upcoming stance phase. As a result, the system can only be treated as a floating base system, with the old stance leg just lifting off the ground and the new stance leg just touching the ground to generate an impulse from the foot. The system dynamics can then be written as

$$\boldsymbol{D}_e(\boldsymbol{q}_e)\ddot{\boldsymbol{q}}_e + \boldsymbol{C}_e(\boldsymbol{q}_e, \dot{\boldsymbol{q}}_e)\dot{\boldsymbol{q}}_e + \boldsymbol{G}_e(\boldsymbol{q}_e) = \boldsymbol{B}_e(\boldsymbol{q}_e)\boldsymbol{u}_e + \delta\boldsymbol{F}_e, \tag{14}$$

where $\boldsymbol{q}_e$ is a column vector containing all system states, defined the same as that of the stance phase, and $\delta\boldsymbol{F}_e$ is the impulsive ground reaction force at the instant of impact. Integrate Equation (14) over the duration of the impact to obtain the additional relationship for this phase,

$$\boldsymbol{D}_e(\boldsymbol{q}_e^+)\dot{\boldsymbol{q}}_e^+ - \boldsymbol{D}_e(\boldsymbol{q}_e^-)\dot{\boldsymbol{q}}_e^- = \boldsymbol{F}_e, \tag{15}$$

where $\boldsymbol{F}_e = \int_{t-}^{t+} \delta\boldsymbol{F}_e(\tau)d\tau$. Note that because the impact is impulsive, $\boldsymbol{q}_e^+ = \boldsymbol{q}_e^-$, and the integration of the other terms in Equation (14) results in zero. Obviously, Equation (15) also has more unknowns than the number of equality constraints. $\dot{\boldsymbol{q}}_e^+$ and $\boldsymbol{F}_e$ in total have $2N + 2L$ unknowns, whereas the number of equality constraints is $N + L$. Similarly, $\boldsymbol{F}_e$ can be further related to the ground reaction force at the new stance foot

$$\boldsymbol{F}_e = \boldsymbol{E}^T\boldsymbol{F}_f. \tag{16}$$

where

$$\boldsymbol{E} = [\frac{\partial\gamma_f(\boldsymbol{q}_s)}{\partial\boldsymbol{q}_s}, \boldsymbol{I}_{L\times L}], \tag{17}$$

Equation (15) can then be converted to

$$\boldsymbol{D}_e(\boldsymbol{q}_e^+)\dot{\boldsymbol{q}}_e^+ - \boldsymbol{D}_e(\boldsymbol{q}_e^-)\dot{\boldsymbol{q}}_e^- = \boldsymbol{E}^T\boldsymbol{F}_f, \tag{18}$$

which now has $N + 2L$ unknowns. Also, because the new stance foot should not slip or rebound, we can have additional L equations in the form

$$\boldsymbol{E}\dot{\boldsymbol{q}}_e^+ = \boldsymbol{0} \tag{19}$$

Combining Equations (18) and (19), the Equations of Motion for the impact phase can be written as

$$\begin{bmatrix} \boldsymbol{D}_e & -\boldsymbol{E}^T \\ \boldsymbol{E} & \boldsymbol{0}_{L\times L} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}_e^+ \\ \boldsymbol{F}_f \end{bmatrix} = \begin{bmatrix} \boldsymbol{D}_e\dot{\boldsymbol{q}}_e^- \\ \boldsymbol{0}_{L\times L} \end{bmatrix} = \begin{bmatrix} \boldsymbol{D}_e\dfrac{\partial\boldsymbol{q}_e}{\partial\boldsymbol{q}_s} \\ \boldsymbol{0}_{L\times N} \end{bmatrix} \dot{\boldsymbol{q}}_s^-, \tag{20}$$

where

$$\frac{\partial\boldsymbol{q}_e}{\partial\boldsymbol{q}_s} = \begin{bmatrix} \boldsymbol{I}_{N\times N} \\ \dfrac{\partial\boldsymbol{p}_e}{\partial\boldsymbol{q}_s} \end{bmatrix}. \tag{21}$$

Solving Equation (20) yields

$$\begin{aligned} &\begin{cases} \dot{\boldsymbol{q}}_e^+ = \boldsymbol{D}_e^{-1}\boldsymbol{E}^T\boldsymbol{F}_f + \dfrac{\partial\boldsymbol{q}_e}{\partial\boldsymbol{q}_s}\dot{\boldsymbol{q}}_s^- \\ \boldsymbol{E}\dot{\boldsymbol{q}}_e^+ = \boldsymbol{0}_{L\times 1} \end{cases} \\ \Rightarrow &\boldsymbol{E}(\boldsymbol{D}_e^{-1}\boldsymbol{E}^T\boldsymbol{F}_f + \frac{\partial\boldsymbol{q}_e}{\partial\boldsymbol{q}_s}\dot{\boldsymbol{q}}_s^-) = \boldsymbol{0}_{L\times 1} \\ \Rightarrow &\begin{cases} \dot{\boldsymbol{q}}_e^+ = [\boldsymbol{I} - \boldsymbol{D}_e^{-1}\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{D}_e^{-1}\boldsymbol{E}^T)^{-1}\boldsymbol{E}]\dfrac{\partial\boldsymbol{q}_e}{\partial\boldsymbol{q}_s}\dot{\boldsymbol{q}}_s^- \\ \boldsymbol{F}_f = -(\boldsymbol{E}\boldsymbol{D}_e^{-1}\boldsymbol{E}^T)^{-1}\boldsymbol{E}\dfrac{\partial\boldsymbol{q}_e}{\partial\boldsymbol{q}_s}\dot{\boldsymbol{q}}_s^- \end{cases} \end{aligned} \tag{22}$$

To complete the Equations of Motion for the impact phase, a relabeling matrix $\boldsymbol{R}$ is needed to automatically swap the stance and swing legs after impact and thus the system get re-initialized for the next stance phase. The relabeling matrix should satisfy the property $\boldsymbol{RR} = \boldsymbol{I}$. As a result, the impact dynamics can be finally written in state space as

$$\boldsymbol{x}^+ = \Delta(\boldsymbol{x}^-) \Rightarrow \begin{bmatrix} \boldsymbol{q}_s^+ \\ \dot{\boldsymbol{q}}_s^+ \end{bmatrix} = \begin{bmatrix} \Delta_{\boldsymbol{q}_s} \boldsymbol{q}_s^- \\ \Delta_{\dot{\boldsymbol{q}}_s} \dot{\boldsymbol{q}}_s^- \end{bmatrix}, \tag{23}$$

where $\Delta_{\boldsymbol{q}_s}$ and $\Delta_{\dot{\boldsymbol{q}}_s}$ are defined as

$$\Delta_{\boldsymbol{q}_s} = \boldsymbol{R}, \tag{24}$$

and

$$\Delta_{\dot{\boldsymbol{q}}_s} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{0}_{N \times L} \end{bmatrix} \frac{\dot{\boldsymbol{q}}_e^+}{\dot{\boldsymbol{q}}_s^-} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{0}_{N \times L} \end{bmatrix} [\boldsymbol{I} - \boldsymbol{D}_e^{-1} \boldsymbol{E}^T (\boldsymbol{E} \boldsymbol{D}_e^{-1} \boldsymbol{E}^T)^{-1} \boldsymbol{E}] \frac{\partial \boldsymbol{q}_e}{\partial \boldsymbol{q}_s}, \tag{25}$$

3 FLIGHT PHASE

To be continued...