Hybrid Linear Inverted Pendulum

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The Hybrid Linear Inverted Pendulum (H-LIP) model adds continuous double support phase (DSP) to replace the classic LIP model's instantaneous impact phase. Point mass and constant Center of Mass (CoM) height are still assumed.

For the single support phase (SSP), the Equations of Motion (EoM) can be written as

$$ {}_S\ddot{x} = \frac{g}{H}\,{}_Sx = \lambda^2\,{}_Sx, \tag{1} $$

where $\lambda = \sqrt{g/H}$ and H is the constant CoM height, while for DSP, the EoM is written as

$$ {}_D\ddot{x} = 0. \tag{2} $$

The transitions from SSP to DSP, $\Delta_{\mathrm{SSP}\to\mathrm{DSP}}$, and from DSP to SSP, $\Delta_{\mathrm{DSP}\to\mathrm{SSP}}$, are assumed to be smooth, and thus defined as

$$ \begin{aligned} \Delta_{\mathrm{SSP}\to\mathrm{DSP}} &: \begin{cases} {}_Dx_i^+ = {}_Sx_i^- \\ {}_D\dot{x}_i^+ = {}_S\dot{x}_i^- \end{cases} \\ \Delta_{\mathrm{DSP}\to\mathrm{SSP}} &: \begin{cases} {}_Sx_{i+1}^+ = {}_Dx_i^- - l_i \\ {}_S\dot{x}_{i+1}^+ = {}_D\dot{x}_i^- \end{cases} \end{aligned} \tag{3} $$

where the superscripts $+$ and $-$ indicate the beginning and ending moment of a phase, l_i is the distance between the two consecutive contact points, and i and $i+1$ are the step indices, or equivalently, the representation of different support legs, which are omitted in the remaining notes when all variables in the equation are of the same step. The closed-form solution can be derived as

$$ \begin{aligned} \mathrm{SSP} &: \begin{cases} {}_Sx(t) = {}_Sx^+ \cosh[\lambda(t - {}_St^+)] + \dfrac{{}_S\dot{x}^+}{\lambda} \sinh[\lambda(t - {}_St^+)] \\ {}_S\dot{x}(t) = {}_Sx^+ \lambda \sinh[\lambda(t - {}_St^+)] + {}_S\dot{x}^+ \cosh[\lambda(t - {}_St^+)] \end{cases} \\ \mathrm{DSP} &: \begin{cases} {}_Dx(t) = {}_Dx^+ + {}_D\dot{x}^+(t - {}_Dt^+) \\ {}_D\dot{x}(t) = {}_D\dot{x}^+ \end{cases} \end{aligned} \tag{4} $$

Due to the characteristic roots being $\pm\lambda$, the phase portrait of SSP (linear autonomous system) is divided into four regions by the two straight lines $\dot{x} = \pm\lambda x$. Note that in the above derivation, the variable x can be replaced by y to represent dynamics in the frontal plane instead of the sagittal plane.

From (3), it can be seen that the horizontal velocity is reserved in phase transitions by the reset map. Therefore, to find period-1 gaits, we can draw a horizontal phase transition line in the phase portrait to mark the desired transition velocity. Intuitively, the system should flow from one point on this transition line to another point on the same line within the SSP, and move back to the starting point along this transition line during the DSP. Obviously, these two points have to be in the top or the bottom region at the same time. Hence, with ${}_S\dot{x}^+ = {}_S\dot{x}^-$, we can have

$$
\begin{aligned}
{}_S\dot{x}^+ &= {}_S\dot{x}^- = {}_Sx^+\lambda\sinh(\lambda\,{}_ST) + {}_S\dot{x}^+\cosh(\lambda\,{}_ST) \\
\Rightarrow \frac{{}_S\dot{x}^+}{{}_Sx^+} &= \frac{\lambda\sinh(\lambda\,{}_ST)}{1-\cosh(\lambda\,{}_ST)} = \lambda\frac{e^{\lambda\,{}_ST}-e^{-\lambda\,{}_ST}}{2-e^{\lambda\,{}_ST}-e^{-\lambda\,{}_ST}} \\
&= -\lambda\frac{(e^{\lambda\,{}_ST/2}-e^{-\lambda\,{}_ST/2})(e^{\lambda\,{}_ST/2}+e^{-\lambda\,{}_ST/2})}{(e^{\lambda\,{}_ST/2}-e^{-\lambda\,{}_ST/2})^2} \\
&= -\lambda\frac{e^{\lambda\,{}_ST/2}+e^{-\lambda\,{}_ST/2}}{e^{\lambda\,{}_ST/2}-e^{-\lambda\,{}_ST/2}} = -\lambda\frac{\cosh(\frac{\lambda\,{}_ST}{2})}{\sinh(\frac{\lambda\,{}_ST}{2})} = -\lambda\coth(\frac{\lambda\,{}_ST}{2})
\end{aligned}
\tag{5}
$$

where ${}_ST = {}_St^- - {}_St^+$. Set $\sigma_1 = \lambda\coth(\frac{\lambda\,{}_ST}{2}) > \lambda$, then the initial and final states, $({}_Sx^+,\ {}_S\dot{x}^+)$ and $({}_Sx^-,\ {}_S\dot{x}^-)$, are on the line $\dot{x} = -\sigma_1 x$ and the line $\dot{x} = \sigma_1 x$, respectively, *i.e.* ${}_S\dot{x}^+ = -\sigma_1\,{}_Sx^+$ and ${}_S\dot{x}^- = \sigma_1\,{}_Sx^-$. Note that for each set of $\dot{x} = \pm\sigma_1 x$, *i.e.* each value of ${}_ST$, there are infinite gaits connected by these two oblique lines. Each such gait is a period-1 gait and has a different value of average forward velocity $\bar{\dot{x}}$ that can be calculated as

$$
\bar{\dot{x}} = \frac{l}{{}_ST + {}_DT} = \frac{({}_Sx^- - {}_Sx^+) + {}_D\dot{x}^+\,{}_DT}{{}_ST + {}_DT} = \frac{{}_S\dot{x}^-(\frac{2}{\sigma_1} + {}_DT)}{{}_ST + {}_DT}.
\tag{6}
$$

where l is the step length. Note that these gaits are symmetric about the $\dot{x}$ axis.

Similarly, to find period-2 gaits, we can draw two horizontal phase transition lines in the phase portrait to mark the two desired transition velocities. Without losing generality, label the transition line with the starting point as Line 1 and the other line as Line 2. Intuitively, the system should flow from a starting point on Line 1 to a point on Line 2 during the first SSP, move along Line 2 to another point for the first DSP, flow from Line 2 back to Line 1 during the second SSP and move along Line 1 back to the starting point during the second DSP. Obviously, each flow should happen in a single region, but the two flows can happen in different regions, which can yield diverse period-2 gaits. Consider the first SSP in a period-2 gait flowing from Line 1 to Line 2, for instance the SSP with left leg as the support leg, we can have

$$
\begin{aligned}
{}_S\dot{y}_L^- &= {}_Sy_L^+\lambda\sinh(\lambda\,{}_ST_L) + {}_S\dot{y}_L^+\cosh(\lambda\,{}_ST_L) \\
&= {}_Sy_L^+\lambda\sinh + {}_S\dot{y}_L^+\cosh \\
&\Rightarrow \\
{}_Sy_L^+ &= \frac{{}_S\dot{y}_L^- - {}_S\dot{y}_L^+\cosh}{\lambda\sinh} \\
&\Rightarrow \\
{}_Sy_L^+ &= \frac{(-\,{}_S\dot{y}_L^+ + \delta) - {}_S\dot{y}_L^+\cosh}{\lambda\sinh} \\
&\Rightarrow \\
{}_S\dot{y}_L^+ &= -\,{}_Sy_L^+\frac{\lambda\sinh}{1+\cosh} + \frac{\delta}{1+\cosh} \\
&\Rightarrow \\
{}_S\dot{y}_L^+ &= -\,\lambda\tanh(\frac{\lambda\,{}_ST_L}{2})\,{}_Sy_L^+ + \frac{\delta}{1+\cosh(\lambda\,{}_ST_L)}
\end{aligned}
\tag{7}
$$

where ${}_S\dot{y}_L^- = -\,{}_S\dot{y}_L^+ + \delta$ is based on symmetry of the special case and δ is an offset added to indicate infinite possible positions of Line 2. Set $\sigma_{2L} = \lambda\tanh(\frac{\lambda\,{}_ST_L}{2}) < \lambda$, then the initial state, $({}_Sy_L^+,\ {}_S\dot{y}_L^+)$, is on the line $\dot{y} = -\sigma_{2L}y + \Delta$, where $\Delta = \delta/(1+\cosh)$. Similarly, we can get that the final state, $({}_Sy_L^-,\ {}_S\dot{y}_L^-)$, is on the line $\dot{y} = \sigma_{2L}y + \Delta$.

Theoretically, the other SSP can be any flow from Line 2 to Line 1. To reduce the number of candidates, we can add a further constraint by setting equal SSP time ${}_ST = {}_ST_L = {}_ST_R$ in the desired period-2 gaits. With ${}_S\dot{y}_L^+ = {}_S\dot{y}_R^-$ and ${}_S\dot{y}_L^- = {}_S\dot{y}_R^+$, we can get

$$
\begin{aligned}
&\begin{cases} {}_S\dot{y}_L^- = {}_Sy_L^+\lambda\sinh(\lambda\,{}_ST) + {}_S\dot{y}_L^+\cosh(\lambda\,{}_ST) \\ {}_S\dot{y}_R^- = {}_Sy_R^+\lambda\sinh(\lambda\,{}_ST) + {}_S\dot{y}_R^+\cosh(\lambda\,{}_ST) \end{cases}
&&\Rightarrow \begin{cases} {}_S\dot{y}_L^- = {}_Sy_L^+\lambda\sinh + {}_S\dot{y}_L^+\cosh \\ {}_S\dot{y}_L^+ = {}_Sy_R^+\lambda\sinh + {}_S\dot{y}_L^-\cosh \end{cases} \\
& &&\Rightarrow \begin{cases} -\,{}_S\dot{y}_L^+\cosh + {}_S\dot{y}_L^- = {}_Sy_L^+\lambda\sinh \\ {}_S\dot{y}_L^+ - {}_S\dot{y}_L^-\cosh = {}_Sy_R^+\lambda\sinh \end{cases} \\
& &&\Rightarrow \begin{cases} -\,{}_S\dot{y}_L^+\cosh + {}_S\dot{y}_L^- = \dfrac{{}_Sy_L^- - \frac{{}_S\dot{y}_L^+}{\lambda}\sinh}{\cosh}\lambda\sinh \\ {}_S\dot{y}_L^+ - {}_S\dot{y}_L^-\cosh = {}_Sy_R^+\lambda\sinh \end{cases} \\
& &&\Rightarrow \begin{cases} {}_S\dot{y}_L^+\dfrac{\sinh^2-\cosh^2}{\cosh} + {}_S\dot{y}_L^- = {}_Sy_L^-\dfrac{\lambda\sinh}{\cosh} \\ {}_S\dot{y}_L^+ - {}_S\dot{y}_L^-\cosh = {}_Sy_R^+\lambda\sinh \end{cases} \\
& &&\Rightarrow \begin{cases} -\,{}_S\dot{y}_L^+ + {}_S\dot{y}_L^-\cosh = {}_Sy_L^-\lambda\sinh \\ {}_S\dot{y}_L^+ - {}_S\dot{y}_L^-\cosh = {}_Sy_R^+\lambda\sinh \end{cases} \\
& &&\Rightarrow {}_Sy_R^+ = -\,{}_Sy_L^-. \\
& &&\Rightarrow \text{Similarly, } {}_Sy_L^+ = -\,{}_Sy_R^-.
\end{aligned}
\tag{8}
$$

The above derivation indicates that to have two SSPs with equal period ${}_ST$ in a period-2 gait, the gait has to be symmetric about the $\dot{y}$ axis. In this case, $\sigma_{2L} = \sigma_{2R} = \sigma_2$. Note that for each value of ${}_ST$, there are infinite gaits connected by the straight line $\dot{y} = -\sigma_2 y + \Delta$. These gaits can be divided into two groups by the intersection point between the line $\dot{y} = -\sigma_2 y + \Delta$ and the line $\dot{y} = -\lambda y$.