Lagrangian to Equations of Motion

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Define the state vector $\boldsymbol{q}$ as an $N \times 1$ column vector

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_N \end{bmatrix}, \tag{1}$$

where N represents the number of generalized coordinates. The derivative of $\boldsymbol{q}$ can be written as

$$\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \vdots \\ \dot{q}_N \end{bmatrix}. \tag{2}$$

Define the Lagrangian as L, the system kinetic energy as K, and the potential energy as V, which are all scalars. Thus, we can have

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = K(\boldsymbol{q}, \dot{\boldsymbol{q}}) - V(\boldsymbol{q}). \tag{3}$$

According to the Lagrange's equation, the following equations need to be derived first,

$$\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}, \tag{4}$$

$$\frac{\partial L}{\partial \boldsymbol{q}} = \frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \boldsymbol{q}} - \frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}, \tag{5}$$

which can then be combined without generalized forces as

$$\begin{aligned} \boldsymbol{0}_{N\times 1} &= \frac{d}{dt}(\frac{\partial L}{\partial \dot{\boldsymbol{q}}})^T - (\frac{\partial L}{\partial \boldsymbol{q}})^T \\ &= \frac{d}{dt}[\frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}]^T - [\frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \boldsymbol{q}}]^T + [\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}]^T \\ &= \frac{\partial}{\partial \dot{\boldsymbol{q}}}[\frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}]^T \frac{d\dot{\boldsymbol{q}}}{dt} + \frac{\partial}{\partial \boldsymbol{q}}[\frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}]^T \frac{d\boldsymbol{q}}{dt} - [\frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \boldsymbol{q}}]^T + [\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}]^T \\ &= \frac{\partial}{\partial \dot{\boldsymbol{q}}}[\frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}]^T \ddot{\boldsymbol{q}} + \frac{\partial}{\partial \boldsymbol{q}}[\frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}]^T \dot{\boldsymbol{q}} - [\frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \boldsymbol{q}}]^T + [\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}]^T. \end{aligned} \tag{6}$$

Set

$$\boldsymbol{D}(\boldsymbol{q}) = \frac{\partial}{\partial \dot{\boldsymbol{q}}}[\frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}]^T, \tag{7}$$

which can be integrated once to get

$$\boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}} = [\frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}]^T, \tag{8}$$

i.e.

$$\frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}} = [\boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}}]^T = \dot{\boldsymbol{q}}^T[\boldsymbol{D}(\boldsymbol{q})]^T, \tag{9}$$

and be integrated once again to get

$$K(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T[\boldsymbol{D}(\boldsymbol{q})]^T\dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{10}$$

Note that $\boldsymbol{D}(q)$ is symmetric, *i.e.* $\boldsymbol{D}(\boldsymbol{q}) = \boldsymbol{D}^T(\boldsymbol{q})$ (Refer to vector differentiation notes for proof inspiration). Therefore, Equation (6) can be further written as

$$\begin{aligned}
\boldsymbol{0}_{N\times 1} &= \boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{\partial[\boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}}]}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} - \{\frac{\partial[\frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}}]}{\partial \boldsymbol{q}}\}^T + [\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}]^T \\
&= \boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{\partial[\boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}}]}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} - \frac{1}{2}[\dot{\boldsymbol{q}}^T\frac{\partial \boldsymbol{D}(\boldsymbol{q})}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}}]^T + [\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}]^T \\
&= \boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{\partial[\boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}}]}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} - \frac{1}{2}[\frac{\partial \boldsymbol{D}(\boldsymbol{q})}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}}]^T\dot{\boldsymbol{q}} + [\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}]^T \\
&= \boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \{\frac{\partial[\boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}}]}{\partial \boldsymbol{q}} - \frac{1}{2}[\frac{\partial \boldsymbol{D}(\boldsymbol{q})}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}}]^T\}\dot{\boldsymbol{q}} + [\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}]^T
\end{aligned}. \tag{11}$$

As a result, the Equations of Motion can be written as

$$\boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{0}_{N\times 1}, \tag{12}$$

where

$$\boldsymbol{D}(\boldsymbol{q}) = \frac{\partial}{\partial \dot{\boldsymbol{q}}}[\frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}]^T,$$

$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{\partial[\boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}}]}{\partial \boldsymbol{q}} - \frac{1}{2}[\frac{\partial \boldsymbol{D}(\boldsymbol{q})}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}}]^T, \tag{13}$$

$$\boldsymbol{G}(\boldsymbol{q}) = [\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}]^T. \tag{14}$$

To look further using indicial notation, we can get

$$D_{ik} = \frac{\partial^2 K}{\partial \dot{q}_i \partial \dot{q}_k}, \tag{15}$$

$$\begin{aligned}
C_{ik} &= \frac{\partial(D_{ij}\dot{q}_j)}{\partial q_k} - \frac{1}{2}[\frac{\partial D_{ij}}{\partial q_k}\dot{q}_j]^T \\
&= \frac{\partial(D_{ij}\dot{q}_j)}{\partial q_k} - \frac{1}{2}\frac{\partial D_{kj}}{\partial q_i}\dot{q}_j \\
&= (\frac{\partial D_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial D_{kj}}{\partial q_i})\dot{q}_j
\end{aligned}. \tag{16}$$

$$G_i = \frac{\partial V}{\partial q_i} \tag{17}$$

$$D_{ik}\ddot{q}_k + C_{ik}\dot{q}_k + G_i = 0 \tag{18}$$