Linear Inverted Pendulum

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The Linear Inverted Pendulum (LIP) model can be used to describe a point mass moving in the sagittal plane via the support of a massless leg. It is commonly used in legged locomotion control as a template . Its Equations of Motion (EoM) can be derived from the Lagrange's equations or the Euler's equations. Note that only the sagittal plane movement is under consideration here, while movement in the frontal plane can be taken care of similarly.

If the Lagrange's equations are used, the Lagrangian can be written as

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{z}^2 - mgz \tag{1}$$

where $\mathcal{L}$ is the Lagrangian, T represents the kinetic energy, V represents the potential energy, m is the system mass, and x, z, $\dot{x}$ and $\dot{z}$ are the horizontal and vertical positions and velocities of the Center of Mass (CoM) with respect to the ground contact point, respectively. According to the Lagrange's equations, we can get

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\partial \mathcal{L}}{\partial \dot{x}} - \dfrac{\partial \mathcal{L}}{\partial x} = F_x \\ \dfrac{\partial}{\partial t}\dfrac{\partial \mathcal{L}}{\partial \dot{z}} - \dfrac{\partial \mathcal{L}}{\partial z} = F_z \end{cases} \Rightarrow \begin{cases} m\ddot{x} = F_x \\ m\ddot{z} + mg = F_z \end{cases} \tag{2}$$

where F_x and F_z are support forces provided by the massless leg to the point mass. Because the resultant force can only be along the leg length direction, we have $F_x z = F_z x$. Therefore, the EoM can be simplified to be

$$\ddot{x} = \frac{g + \ddot{z}}{z}x \tag{3}$$

If the CoM height is assumed to be constant, *i.e.* $z = H$ and $\dot{z} = \ddot{z} = 0$, a more simplified model can be generated as

$$\ddot{x} = \frac{g}{H}x \tag{4}$$

If the Euler's equations are used, we can have

$$\dot{L} = mgx \tag{5}$$

where L is the angular momentum about the ground contact point. On the other hand, we can write the angular momentum based on its definition as

$$L = L_c + mx\dot{z} - mz\dot{x} \tag{6}$$

where L_c is the angular momentum about the Center of Mass (CoM). Note that the point mass assumption is not necessary here. Take the derivative with respect to time on both sides of (6), we can have

$$\dot{L} = \dot{L}_c + mx\ddot{z} - mz\ddot{x} \tag{7}$$

Combining (5) and (7) to cancel $\dot{L}$, we can have a model similar to the one from the Lagrange's equations, whose state space expression can be written as

$$\begin{cases} \dot{x} = v, \\ \dot{v} = \dfrac{g + \ddot{z}}{z} x - \dfrac{\dot{L}_c}{mz}, \end{cases} \tag{8}$$

where v is the system's horizontal velocity. From another perspective, we can also combine (5) and (6) to get a model in state space as

$$\begin{cases} \dot{x} = \dfrac{L}{mz} + \dfrac{\dot{z}}{z} x - \dfrac{L_c}{mz}, \\ \dot{L} = mgx, \end{cases} \tag{9}$$

Note that (8) and (9) are equivalent at this point, *i.e.* different forms of expressions about the same system.

Now let us start to make assumptions to get the reduced-order models. If the CoM height is assumed to be constant, *i.e* $z = H$ and $\dot{z} = \ddot{z} = 0$, we can have

$$\begin{cases} \dot{x} = v, \\ \dot{v} = \dfrac{g}{H} x - \dfrac{\dot{L}_c}{mH}, \end{cases} \tag{10}$$

and

$$\begin{cases} \dot{x} = \dfrac{L}{mH} - \dfrac{L_c}{mH}, \\ \dot{L} = mgx, \end{cases} \quad \text{or} \quad \begin{cases} \dot{x} = v_s - \dfrac{L_c}{mH}, \\ \dot{v}_s = \dfrac{g}{H} x, \end{cases} \tag{11}$$

where $v_s = L/(mH)$ is a surrogate variable for the convenience of comparison between (10) and (11). Note that at this point, (10) and (11) are still equivalent as long as the CoM height is constant. If the body mass is further assumed to be point mass, *i.e.* $L_c = 0$, we can have

$$\begin{cases} \dot{x} = v, \\ \dot{v} = \dfrac{g}{H} x, \end{cases} \tag{12}$$

and

$$\begin{cases} \dot{x} = \dfrac{L}{mH}, \\ \dot{L} = mgx, \end{cases} \tag{13}$$

where (12) is the same as (4), which is the well-known LIP model that the literatures commonly refer to, and (13) is called Angular momentum LIP (ALIP) model.

A further remark is that if $L_c = 0$ is not true, which is generally the case with real robot, (12) and (13) are two different forms of approximation. However, (12) ignores the subtraction of $\dot{L}_c$ from mgx and (13) ignores the subtraction of L_c from L, while $\dot{L}_c/mgx$ is much larger than L_c/L over a wide range of walking velocities based on literature. Therefore, (13) has relatively less error. On the other hand, in (10), v has a relative degree of one with respect to $\dot{L}_c$, while in (11), L has a relative degree of three with respect to $\dot{L}_c$. Hence, v is more sensitive than L to the omission of L_c. Again, (13) is a better approximation than (12).

The LIP model in (12) can be analytically solved for a closed-form solution. Note that only the homogeneous form is considered here, which is

$$\ddot{x} = \lambda^2 x \tag{14}$$

with $\lambda = \sqrt{g/H}$, while actuation at the contact point can also be added to yield the inhomogeneous form. The corresponding characteristic function of the homogeneous form is $s^2 = \lambda^2$, whose roots are $s_1 = -\lambda$

and $s_2 = \lambda$. Therefore, the general solution can be written as

$$\begin{cases} x(t) = C_1 e^{\lambda t} + C_2 e^{-\lambda t}, \\ \dot{x}(t) = C_1 \lambda e^{\lambda t} - C_2 \lambda e^{-\lambda t}, \end{cases} \tag{15}$$

where C_1 and C_2 are two constant coefficients related to the initial conditions. Set $x(t_o) = x_o$ and $\dot{x}(t_o) = \dot{x}_o$ as the initial conditions, then (15) can further yields

$$\begin{aligned} &\begin{cases} x_o = C_1 e^{\lambda t_o} + C_2 e^{-\lambda t_o}, \\ \dot{x}_o = C_1 \lambda e^{\lambda t_o} - C_2 \lambda e^{-\lambda t_o}, \end{cases} \quad \Rightarrow \begin{cases} C_1 = e^{-\lambda t_o}(x_o + \dot{x}_o/\lambda)/2, \\ C_2 = e^{\lambda t_o}(x_o - \dot{x}_o/\lambda)/2, \end{cases} \\ &\Rightarrow \begin{cases} x(t) = x_o \dfrac{e^{\lambda(t-t_o)} + e^{-\lambda(t-t_o)}}{2} + \dfrac{\dot{x}_o}{\lambda} \dfrac{e^{\lambda(t-t_o)} - e^{-\lambda(t-t_o)}}{2}, \\ \dot{x}(t) = x_o \lambda \dfrac{e^{\lambda(t-t_o)} - e^{-\lambda(t-t_o)}}{2} + \dot{x}_o \dfrac{e^{\lambda(t-t_o)} + e^{-\lambda(t-t_o)}}{2}, \end{cases} \\ &\Rightarrow \begin{cases} x(t) = x_o \cosh[\lambda(t - t_o)] + \dfrac{\dot{x}_o}{\lambda} \sinh[\lambda(t - t_o)], \\ \dot{x}(t) = x_o \lambda \sinh[\lambda(t - t_o)] + \dot{x}_o \cosh[\lambda(t - t_o)]. \end{cases} \end{aligned} \tag{16}$$

At the end of the single support phase, an instantaneous impact phase is usually applied to reset the system for the next step. The reset map can be written as

$$\begin{cases} x^+ = x^- - l \\ \dot{x}^+ = \dot{x}^- \end{cases} \tag{17}$$

where the superscript $+$ and $-$ indicates the beginning and ending moment of the single support phase, respectively, and l represents the step length. Note that the continuity of velocity is assumed here, which is an acceptable assumption but is usually not the case in reality due to the ground impact.